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A Unified Optimization Algorithm for Bang-bang Optimal Control

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This paper proposes a unified algorithm based on iterative second-order cone programming (SOCP) to solve bang-bang optimal control problems. For a bang-bang optimal control problem, the control values are constrained at the upper or lower bound. We first formulate the bang-bang optimal control problem as a nonconvex quadratically constrained quadratic programming (QCQP) problem by expressing the bang-bang control profiles as quadratic equality constraints. Then an iterative algorithm is proposed to solve nonconvex QCQPs, where each iteration is formulated as a SOCP problem. To obtain robust convergence of the proposed iterative algorithm under random initial guess, a multi-stage framework, combined with the relaxation technique, is introduced. To be specific, in the first stage, the QCQP problem is reformulated, where the terminal equality constraints are removed and handled as weighted penalty terms in the objective function. Together with the relaxed bang-bang constraints, the reformulated problem in the first stage is solved via the iterative SOCP and its solution is used as the initial guess for the second stage. In the second stage, the bang-bang constraints of the original problem are considered to obtain the final solution. Finally, the proposed algorithm is applied to the fuel-optimal powered descent guidance problem, and the effectiveness and robustness of the proposed algorithm are verified via numerical simulations.

I. Introduction

This paper investigates a class of optimal control problems (OCPs), where the control profile is restrained to be bang-bang control. Bang-bang control, also called on-off control, refers to the control profile that switches abruptly between the lower bound and the upper bound. In the past decades, bang-bang optimal control has wide application in various engineering fields, such as powered descent guidance [1, 2], entry guidance [3], and collision avoidance control [4]. Assorted algorithms have been developed to solve the bang-bang OCPs. For example, work in [5] searches the optimal switching points for a time-optimal bang-bang control system by allowing the initial states to be optimized along with the time. In [6], the control parametrization enhancing transform method has been developed to calculate the switching times as well as the singular control values. Besides, nonlinear programming has also been applied to solve the bang-bang OCPs [7]. However, there is no unified approach to solve general bang-bang OCPs that guarantee yielding an exact bang-bang control profile while optimizing the desired performance index.

By expressing the bang-bang profile as quadratic equality constraints and employing the discretization techniques, a bang-bang OCP can be reformulated as a nonconvex quadratically constrained quadratic programming (QCQP) problem. The quadratic equality constraints on the bang-bang control profile guarantee that the obtained control solution is an exact bang-bang curve when the quadratic equality constraints are satisfied. QCQP is to minimize a quadratic objective function subject to quadratic equality or inequality constraints, which has been applied in a wide range of optimization problems, including but not limited to radar detection [8], signal processing [9] and path planning [10]. Extensive numerical methods have been developed to solve nonconvex QCQPs, e.g., successive convex approximation (SCA) and relaxation methods [11–13]. By approximating the nonconvex functions by a series of convex surrogates, SCA methods solve a QCQP problem iteratively by formulating each iteration as a convex optimization problem. On the other hand, two typical relaxation methods, reformulation-linearization technique [14] and semi-definite relaxation [15], have also

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been applied to solve QCQPs. The Semi-definite relaxation method can find a lower bound on the objective function, which, however, cannot guarantee an optimal solution or even a feasible one in general cases.

In this paper, we propose a unified optimization scheme using iterative second-order cone programming (SOCP) to solve general bang-bang OCPs. Firstly, the discretization technique is applied to convert a bang-bang OCP into a nonconvex QCQP problem. A QCQP can be converted into a semidefinite programming (SDP) problem with a rank-one constraint on the unknown matrix. Next, combining matrix decomposition, the semidefinite constraints in the rank-one constrained SDP can be replaced with second-order cone constraints, which are represented by convex quadratic inequalities and can be solved more efficiently than SDP in general [16]. By formulating all the constraints as linear or second order cone constraints, a QCQP is solved in a successive manner until it converges, where each iteration is a convex SOCP problem.

To obtain robust convergence of the proposed iterative QCQP algorithm under random initial guess, a multi-stage framework combined with the relaxation technique is introduced. To be specific, in the first stage, the QCQP problem is reformulated, where the terminal equality constraints are removed and handled as weighted penalty terms in the objective function. Together with the relaxation on the bang-bang constraints, the reformulated problem in the first stage is solved via the iterative SOCP and its solution is used as the initial guess for the second stage. In the second stage, the bang-bang constraints of the original problem are considered to obtain the final solution.

To verify effectiveness and robustness of the proposed algorithm, the multi-stage framework, together with the iterative QCQP, is implemented in the fuel-optimal powered descent guidance problem. The quadratic equality constraints are used to express the bang-bang control profile and then the powered descent guidance problem is formulated as a QCQP problem through discretization. Comparative examples from a commercial nonlinear programming solver are provided to demonstrate the computational advantages of the proposed algorithm.

This paper is organized as follows. Section II introduces the problem formulation, where the bang-bang OCP is formulated as a general QCQP. The framework of the proposed multi-stage iterative SOCP method is detailed in Section III. After that, numerical simulations are provided and analyzed in Section IV. Conclusions are addressed in Section V.

II. Problem Formulation

A. Original Bang-Bang OCP

Consider a bang-bang OCP formulated as

$$
\begin{aligned}
\min_{\mathbf{u}(t),t_f} \quad & \mathbf{J} = \phi(\mathbf{x}(t_f),\ t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(t),\mathbf{u}(t))dt, && \text{(1a)}\\
\text{s.t.} \quad & \dot{\mathbf{x}}(t) = f(\mathbf{x}(t),\ \mathbf{u}(t)), && \text{(1b)}\\
& \mathbf{g}_j(\mathbf{x}(t),\mathbf{u}(t)) \leq 0,\ j = 1,2,\dots,g_n, && \text{(1c)}\\
& \mathbf{x}(t_0) = \mathbf{x}_0,\ \ \mathbf{x}(t_f) = \mathbf{x}_f, && \text{(1d)}\\
& \mathbf{u} \in \mathcal{U}, && \text{(1e)}
\end{aligned}
$$

where $\mathbf{x} = [x_1, \cdots, x_{s_1}]^T \in \mathbb{R}^{s_1}$ represents the state vector, and $\mathbf{u} = [u_1, \cdots, u_{s_2}]^T \in \mathbb{R}^{s_2}$ is the control vector. The initial state $\mathbf{x}_0$ at a given time t_0, and the terminal state $\mathbf{x}_f$ are specified. The final time t_f is free. Furthermore, $\mathbf{g}_j(\mathbf{x}(t),\mathbf{u}(t)) \leq 0$ represent the state and control constraints and g_n is the number of the inequality constraints. The control vector in (1e) is restrained by bang-bang constraints, expressed as

$$
\mathcal{U} = \left\{\mathbf{u}|u_i \in \left\{u_i^{min}, u_i^{max}\right\},\ i = 1,\cdots,s_2\right\}, \tag{2}
$$

where u_i^{min} and u_i^{max} represent the lower bound and the upper bound of the control variable u_i, respectively. The bang-bang control constraint on u_i can be equivalently formulated as a quadratic equality constraint in the form of $(u_i - u_i^{min})(u_i - u_i^{max}) = 0$.

B. The QCQP Formulation of a Bang-Bang OCP

By applying the discretization techniques, problem (1) can be discretized into a finite number of discrete points, represented by discrete state and control variables at each point. Accordingly, the dynamics in (1b) can be rewritten as

$$
\dot{\mathbf{x}} = \frac{\mathbf{x}_{h+1} - \mathbf{x}_h}{\Delta t} = f(\mathbf{x}_h,\ \mathbf{u}_h), \tag{3}
$$

where $\mathbf{x}_h$ denotes the state variables at the hth node, $h = 1, ..., H$, with H being the number of discrete nodes. Additionally, Δt is the duration of the time interval between two adjacent nodes. Combining with (3), problem (1) can be reformulated as

$$\begin{aligned}
\min_{\mathbf{u}(t),\Delta t} \quad & \mathbf{J} = \phi(\mathbf{x}_H,\ \Delta t) + \sum_{h=1}^{H} L(\mathbf{x}_h, \mathbf{u}_h)\Delta t, && \text{(4a)}\\
\text{s.t.} \quad & \frac{\mathbf{x}_{h+1} - \mathbf{x}_h}{\Delta t} = f(\mathbf{x}_h,\ \mathbf{u}_h),\ h = 1, 2, \ldots, H, && \text{(4b)}\\
& \mathbf{g}_j(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_H, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_H) \leq 0,\ j = 1, 2, \ldots, g_n, && \text{(4c)}\\
& \mathbf{x}_1 = \mathbf{x}_0, \quad \mathbf{x}_H = \mathbf{x}_f, && \text{(4d)}\\
& \mathbf{u}_h \in \mathcal{U}, h = 1, 2, \ldots, H. && \text{(4e)}
\end{aligned}$$

In the above problem, constraints and the objective function can be transformed into a quadratic form by introducing new variables associated with quadratic equality constraints, except for those containing non-polynomial functions [17]. In addition, when an equality constraint is considered, it can be equivalently converted into two inequality constraints. Then the discretized problem in (4) can be reformulated as an inhomogeneous QCQP problem, written as

$$\begin{aligned}
\min_{\bar{\mathbf{x}}} \quad & \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}_0 \bar{\mathbf{x}} + \mathbf{b}_0^T \bar{\mathbf{x}} + c_0, \\
\text{s.t.} \quad & \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}_j \bar{\mathbf{x}} + \mathbf{b}_j^T \bar{\mathbf{x}} + c_j \leq 0,\ j = 1, 2, ..., N_i,
\end{aligned} \tag{5}$$

where $\bar{\mathbf{x}} = [\mathbf{x}^T, \mathbf{u}^T] \in \mathbb{R}^m$ is the vector to be optimized, $A_j \in \mathbb{R}^{m\times m}$ are symmetric matrices, $\mathbf{b}_j \in \mathbb{R}^m$ are known vectors and $c_j \in \mathbb{R}$ are given constants, N_i represents the number of the inequality constraints.

III. Methods for Solving QCQP problem

A. The Iterative SOCP Algorithm

First, the inhomogeneous QCQP problem in (5) is reformulated as a homogeneous rank-one constrained problem. Considering the inhomogeneous QCQP (5), by introducing a new variable $\mathbf{z} = [\bar{\mathbf{x}}^T, 1]^T$, problem (5) can be reformulated as a homogeneous QCQP

$$\begin{aligned}
\min_{\mathbf{z}\in\mathbb{R}^{n+1}} \quad & \mathbf{z}^T \mathbf{Q}_0 \mathbf{z}, \\
\text{s.t.} \quad & \mathbf{z}^{\mathrm{T}} \mathbf{Q}_j \mathbf{z} + c_j \leq 0,\ j = 1, ..., n,
\end{aligned} \tag{6}$$

where $\mathbf{Q}_j = \begin{bmatrix} \mathbf{A}_j & \mathbf{b}_j/2 \\ \mathbf{b}_j^T/2 & 0 \end{bmatrix}$. Note that $\mathbf{z}^T\mathbf{Q}\mathbf{z} = \mathbf{Tr}(\mathbf{Q}\mathbf{z}\mathbf{z}^T)$, where $\mathbf{Tr}(\cdot)$ represents the trace of a matrix. By denoting $\mathbf{Z} = \mathbf{z}\mathbf{z}^T \in \mathbb{R}^{(m+1)\times(m+1)}$, problem (6) can be expressed as

$$\begin{aligned}
\min_{\mathbf{Z},\mathbf{z}} \quad & \mathbf{Tr}(\mathbf{Q}_0\mathbf{Z}), \\
\text{s.t.} \quad & \mathbf{Tr}(\mathbf{Q}_j\mathbf{Z}) + c_j \leq 0,\ j = 1, ..., n, \\
& \mathbf{Z} = \mathbf{z}\mathbf{z}^T,
\end{aligned} \tag{7}$$

where item Z_{ij} in matrix $\mathbf{Z}$ equals to $z_i z_j$. For the constraint $\mathbf{Z} = \mathbf{z}\mathbf{z}^T$, it can be equivalently replaced by a rank-one constraint, $\mathrm{rank}(\mathbf{Z}) \leq 1$, associated with a semidefinite constraint $\mathbf{Z} \succeq \mathbf{0}$. Thus, problem (7) can be rewritten as a rank-one constrained SDP problem in the form

$$\begin{aligned}
\min_{\mathbf{Z}} \quad & \mathbf{Tr}(\mathbf{Q}_0\mathbf{Z}), \\
\text{s.t.} \quad & \mathbf{Tr}(\mathbf{Q}_j\mathbf{Z}) + c_j \leq 0,\ j = 1, ..., n, \\
& \mathrm{rank}(\mathbf{Z}) \leq 1, \\
& \mathbf{Z} \succeq \mathbf{0}.
\end{aligned} \tag{8}$$

Therefore, without loss of generality, we equivalently convert a general QCQP problem into a rank-one constrained SDP problem. However, for problem (8), the rank-one constraint is still nonconvex and it is time consuming to solve large-scale SDP problems.

In the second part, to improve the computational efficiency of the algorithm, the semidefinite constraint is replaced by a set of second-order cone constraints. Here we denote a 2×2 sub-matrix of $\mathbf{Z}$ as

$$\mathbf{Z}_\beta = \begin{bmatrix} Z(p,p) & Z(p,q) \\ Z(q,p) & Z(q,q) \end{bmatrix},$$

where β represents an integer pair (p,q). In addition, two sets $\mathcal{F}$ and $\mathcal{G}$ are defined to describe the integer pairs,

$$\mathcal{F} := \{(p,q) | q \neq m+1\},$$

$$\mathcal{G} := \{(p,q) | q = m+1\}.$$

In problem (8), we have $\mathbf{z} = [\bar{\mathbf{x}}^T, 1]^T$ and $\mathbf{Z} = \mathbf{z}\mathbf{z}^T$, which indicates that $Z(m+1, m+1) = 1$. Therefore, for a 2×2 sub-matrix $\mathbf{Z}_\gamma$, where $\gamma \in \mathcal{G}$, we have

$$\mathbf{Z}_\gamma = \begin{bmatrix} Z(p,p) & Z(p,m+1) \\ Z(m+1,p) & 1 \end{bmatrix}.$$

Theorem III.1 *[18] A rank-one symmetric semidefinite matrix has all of its 2×2 primal minors equal to zero, which means all of its 2×2 sub-matrices are positive semidefinite with rank equals to one.*

According to Theorem III.1, for problem (8), the semidefinite constraint $\mathbf{Z}_\gamma \succeq \mathbf{0}$ can be rewritten as

$$\begin{gathered} Z(p,p) \geq 0, \\ Z(p,p) - Z(p,m+1)^2 \geq 0, \end{gathered}$$

which are linear and second-order-cone constraints. It is obvious that, if non-zero entries only exist at the last column, last row and principal diagonal of the coefficient matrices, we only need to consider semidefinite constraints on the sub-matrix $\mathbf{Z}_\gamma$, where $\gamma \in \mathcal{G}$. $Z(p,q)$ can be rewritten as a quadratic function, expressed as

$$Z(p,q) = z(p)z(q) = \frac{1}{2}[(z(p)+z(q))^2 - z(p)^2 - z(q)^2]. \tag{9}$$

By introducing a new vector $\mathbf{y} = [y_1, \cdots, y_h]^T$, where $y_{i_h} = z(p) + z(q)$, $i_h = 1, \ldots, h$, the pair $(p,q) \in \mathcal{F}$, and h is the number of cross terms that are involved in the set $\mathcal{F}$, then (9) can be rewritten as $2Z(p,q) = (y_{i_h})^2 - Z(p,p) - Z(q,q)$. Let $\hat{\mathbf{z}} = [\mathbf{z}^T, \mathbf{y}^T, 1]^T \in \mathbb{R}^{(m+h+2)}$ and $\hat{\mathbf{Z}} = \hat{\mathbf{z}}\hat{\mathbf{z}}^T$, then $2\hat{Z}(p,q) = \hat{Z}(n+i, n+i) - \hat{Z}(p,p) - \hat{Z}(q,q)$, $i_h = 1, \ldots, h$. Since the original unknown vector $\mathbf{z}$ is now extended to $\hat{\mathbf{z}}$, the coefficient matrices $\mathbf{Q}_j$ is now reformulated as $\hat{\mathbf{Q}}_j \in \mathbb{S}^{(m+h+2)\times(m+h+2)}$ such that $\mathbf{Tr}(\mathbf{Q}_j\mathbf{Z}) = \mathbf{Tr}(\hat{\mathbf{Q}}_j\hat{\mathbf{Z}})$, $j = 1, \ldots, n$. Besides, considering that only non-zero entries of $\hat{\mathbf{Q}}_j$, $j = 0, \ldots, n$, will be converted into second-order cone constraints, the following definition is introduced.

Definition III.2 *Let $\hat{\mathbf{Q}}_t = \sum_{j=0}^{m} abs(\hat{\mathbf{Q}}_j)$, where $abs(\hat{\mathbf{Q}}_j)$ represents the element-wise absolute value of the matrix $\hat{\mathbf{Q}}_j$, then we can define a set $\hat{\mathcal{H}}$ as*

$$\hat{\mathcal{H}} := \{(p,q) | p, q \in \{1, ..., m+h+2\} \ \& \ 1 \leq p < q \leq m+h+2\}.$$

Denote the kth entry in $\hat{H}$ as β_k, where $1 \leq k \leq K$, $K = \frac{(m+h+1)(m+h+2)}{2}$. Then set $\hat{\mathcal{K}}$ is defined as

$$\hat{\mathcal{K}} := \{\hat{\beta}_k | \hat{Q}_t(p,q) \neq 0 \ \& \ \hat{\beta}_k \in \hat{\mathcal{H}}\},$$

where $\hat{Q}_t(p,q)$ is the entry in pth row and qth column of matrix $\hat{\mathbf{Q}}_t$. Similarly, we have

$$\hat{\mathcal{F}} := \{\hat{\beta}_f | q \neq m+h+2 \ \& \ \hat{\beta}_f \in \hat{\mathcal{K}}\} = \emptyset,$$

$$\hat{\mathcal{G}} := \{\hat{\beta}_g | q = m+h+2 \ \& \ \hat{\beta}_g \in \hat{\mathcal{K}}\}.$$

With the above definition, the rank-constrained SDP problem in (8) can be rewritten as a SOCP problem with a rank-one constraint,

$$\begin{aligned}
\min_{\hat{\mathbf{Z}}} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_0\hat{\mathbf{Z}}), \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_j\hat{\mathbf{Z}}) + c_j \leq 0,\ j = 1, ..., n, \\
& \hat{\mathbf{Z}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{Z}(p,p) & \hat{Z}(p,q) \\ \hat{Z}(q,p) & \hat{Z}(q,q) \end{bmatrix}, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(m+i_h, 1) = \hat{Z}(p,1) + \hat{Z}(q,1), \forall \hat{\beta}_f \in \hat{\mathcal{F}}, i_h = 1, \dots, h, \\
& \hat{Z}(p,p) \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(p,p) - \hat{Z}(p,n+1)^2 \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \text{rank}(\hat{\mathbf{Z}}) \leq 1.
\end{aligned} \tag{10}$$

Next, according to Theorem III.1, by decomposing the matrix Z, the rank-one constraint can be replaced by multiple rank constraints applying on its 2×2 principle submatrices. Thus, problem (8) can be equivalently converted into a new rank-one constrained SDP problem, expressed as

$$\begin{aligned}
\min_{\hat{\mathbf{Z}}} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_0\hat{\mathbf{Z}}), \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_j\hat{\mathbf{Z}}) + c_j \leq 0,\ j = 1, ..., n, \\
& \hat{\mathbf{Z}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{Z}(p,p) & \hat{Z}(p,q) \\ \hat{Z}(q,p) & \hat{Z}(q,q) \end{bmatrix}, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(m+i_h, 1) = \hat{Z}(p,1) + \hat{Z}(q,1), \forall \hat{\beta}_f \in \mathcal{F}, i_h = 1, \dots, h, \\
& \hat{Z}(p,p) \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(p,p) - \hat{Y}(p,n+1)^2 \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \text{rank}(\hat{\mathbf{Z}}_{\hat{\beta}_{\mathbf{g}}}) \leq 1.
\end{aligned} \tag{11}$$

For each 2×2 rank-one matrix $\hat{\mathbf{Z}}_{\hat{\beta}_g}$, we define λ_1 and λ_2 as its two eigenvalues, and assume $\lambda_1 \leq \lambda_2$. Correspondingly, there are two eigenvectors $\mathbf{v}^1_{\hat{\beta}_g}$ and $\mathbf{v}^2_{\hat{\beta}_g}$ which have

$$\lambda_1 \mathbf{v}^1_{\hat{\beta}_g} = \hat{\mathbf{Z}}_{\hat{\beta}_g}\mathbf{v}^1_{\hat{\beta}_g},\ \lambda_2 \mathbf{v}^2_{\hat{\beta}_g} = \hat{\mathbf{Z}}_{\hat{\beta}_g}\mathbf{v}^2_{\hat{\beta}_g}. \tag{12}$$

Combining the fact that $\hat{\mathbf{Z}}_{\hat{\beta}_g}$ is a rank-one semidefinite matrix, which indicates that $\lambda_2 \geq \lambda_1$ and $\lambda_1 = 0$, we have

$$(\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{Z}}_{\hat{\beta}_g}\mathbf{v}^1_{\hat{\beta}_g} = (\mathbf{v}^1_{\hat{\beta}_g})^T(\lambda_1 \mathbf{v}^1_{\hat{\beta}_g}) = 0. \tag{13}$$

By introducing another independent variable $r_{\hat{\beta}_g} \in \mathbb{R}$, the rank-one constraint $\text{rank}(\hat{\mathbf{Z}}_{\hat{\beta}_g}) = 1$ can be reformulated as

$$r_{\hat{\beta}_g} - (\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{Z}}_{\hat{\beta}_g}\mathbf{v}^1_{\hat{\beta}_g} \geq 0, \tag{14}$$

where $r_{\hat{\beta}_g} = 0$. However, we can not obtain the eigenvectors $\mathbf{v}^1_{\hat{\beta}_g}$ and $\mathbf{v}^2_{\hat{\beta}_g}$ before getting the exact $\hat{\mathbf{Z}}_{\hat{\beta}_g}$.

Finally, in the last part, an iterative method is applied to approach the optimal eigenvectors by gradually reducing the independent variable $r_{\hat{\beta}_g}$. To achieve this goal, a new problem can be formulated to minimize the original objective function, along with the penalty term associated with the rank-one constraint, expressed as

$$
\begin{aligned}
\min_{\hat{\mathbf{Z}}} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_0\hat{\mathbf{Z}}) + \omega_l \sum_{\hat{\beta}_g \in \hat{\mathcal{G}}} r_{\hat{\beta}_g} \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{Q}}_j\hat{\mathbf{Z}}) + c_j \leq 0, \quad j = 1, ..., n, \\
& \hat{\mathbf{Z}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{Z}(p,p) & \hat{Z}(p,q) \\ \hat{Z}(q,p) & \hat{Z}(q,q) \end{bmatrix}, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(m+i_h, 1) = \hat{Z}(p,1) + \hat{Z}(q,1), \forall \hat{\beta}_f \in \mathcal{F}, \ i_h = 1, \dots, h, \\
& \hat{Z}(p,p) \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{Z}(p,p) - \hat{Z}(p,n+1)^2 \geq 0, \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& r_{\hat{\beta}_g} - (\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{Z}}_{\hat{\beta}_g} \mathbf{v}^1_{\hat{\beta}_g} \geq 0, \ \forall \hat{\beta}_g \in \hat{\mathcal{G}},
\end{aligned}
\tag{15}
$$

where $\omega_l > 0$ is the weighting factor for $\hat{\beta}_g$ at the lth iteration. However, before finding $\hat{\mathbf{Z}}_{\hat{\beta}_g}$ for all $\hat{\beta}_g \in \hat{\mathcal{G}}$, we cannot obtain its corresponding eigenvectors, $\mathbf{v}^1_{\hat{\beta}_g}$. Therefore, an iterative SOCP algorithm is proposed to solve the QCQP problem. The steps of the iterative SOCP are listed in Table 1.

Table 1 Flowchart of the Iterative SOCP Algorithm

Input: $\mathbf{A}_j$, $\mathbf{b}_j$, c_j, $j = 0, 1, ..., n$, ω_l, $\hat{l}_{\max}$, and ϵ
Output: Unknown vector $\mathbf{x}$
begin:
1) Calculate $\hat{\mathbf{Q}}_j$, $j = 0, \dots n$, according to the input
2) Compute $\hat{\mathbf{Z}}$ and $\mathbf{v}^1_{\hat{\beta}_g}$ with a random initial guess
3) **for** $l = 1, 2, ..., \hat{l}_{\max}$
4) Solve (15) to obtain solution $\hat{\mathbf{Z}}$ and $r_{\hat{\beta}_g}$,
5) **If** $\sum_{\hat{\beta}_g \in \hat{\mathcal{G}}} r_{\hat{\beta}_g} \leq \epsilon$,
6) break;
7) **else**
8) Update $\mathbf{v}^1_{\hat{\beta}_g}$ from eigenvectors of $\hat{\mathbf{Z}}_{\beta_g}$
9) **end if**
10) $l = l + 1$
11) **end for**

B. Multi-stage Iterative Algorithm

To obtain robust convergence of the proposed iterative algorithm under random initial guess, a multi-stage iterative algorithm is proposed, which includes two stages. In the first stage, the terminal boundary constraints are relaxed. To be specific, when solving the bang-bang OCP, the initial guess of states are generated by integrating the dynamics with random control variables and a random integration time step. The terminal values obtained from integrating the dynamics with random control cannot satisfy the boundary constraints in general cases. These terminal boundary constraints are relaxed by introducing slack variables ζ_k, $k = 1, \dots, s_1$, that are handled as weighted penalty terms in the augmented objective function. The quadratic equality constraints on the bang-bang control profile are also relaxed as upper and lowered bound constraints in the first stage. Accordingly, problem in (5) is reformulated as

$$\min_{\bar{\mathbf{x}}} \quad \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}_0 \bar{\mathbf{x}} + \mathbf{b}_0^T \bar{\mathbf{x}} + c_0 + \sum_{k=1}^{s_1} \mu_k \cdot \zeta_k^2 \tag{16a}$$

$$\text{s.t.} \quad \frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}_j \bar{\mathbf{x}} + \mathbf{b}_j^T \bar{\mathbf{x}} + c_j \leq 0, \ j = 1, 2, ..., N_i - 2s_1. \tag{16b}$$

$$\frac{1}{2}\bar{\mathbf{x}}^T \mathbf{A}_{k+N_i-s_1} \bar{\mathbf{x}} + \mathbf{b}_{k+N_i-s_1}^T \bar{\mathbf{x}} + c_{k+N_i-s_1} - \zeta_k = 0, \ k = 1, 2, ..., s_1. \tag{16c}$$

where (16c) represents the terminal boundary constraints. The relaxed problem in the first stage is then solved by the iterative SOCP algorithm.

For the second stage, the bang-bang constraints are reconsidered. The solution from the first stage is regarded as the initial guess of the second stage. Using the iterative SOCP algorithm again, the formulated problem in the second stage, which includes all constraints of the original problem, is resolved. The framework of the proposed multi-stage iterative algorithm is presented in Table 2.

Table 2 Flowchart of the Multi-stage Iterative Algorithm

STAGE 1
Input: $\mathbf{A}_j$, $\mathbf{b}_j$, c_j, $j = 0, 1, ..., n$, random initial guess $\bar{\mathbf{x}}^0$, ω_l, $\hat{l}_{\max}$, and ϵ
Output: Unknown vector $\hat{\mathbf{Z}}$
Begin: Reformulate the original QCQP problem (5) into relaxed problem (16)
according to the initial guess $\bar{\mathbf{x}}^0$, and apply the iterative SOCP algorithm to solve it,
until the converged $\hat{\mathbf{Z}}$ is obtained.
STAGE 2
Input: The reformulated problem (16), bang-bang constraints (4e), ω_l, $\hat{l}_{\max}$, and ϵ
$\hat{\mathbf{Z}}$ from STAGE 1 as the initial guess,
Output: Unknown vector $\hat{\mathbf{Z}}$
Begin: Apply the iterative SOCP algorithm to solve (16) with bang-bang constraints (4e),
until the converged $\hat{\mathbf{Z}}$ is obtained.

IV. Simulation Results

To verify the performance of the proposed algorithm, the two-degree-of-freedom (2-DoF) fuel-optimal powered descent guidance (FOPDG) problem [1], which is a representative application of the bang-bang optimal control, is solved by the proposed algorithm. In addition, comparative results from a commercial nonlinear-programming (NLP) solver [19] are provided.

For the FOPDG problem, the dynamics of a vehicle equipped with a retro-propulsion system, in a Cartesian coordinate, is given by

$$\dot{\mathbf{r}} = \mathbf{v}, \tag{17a}$$

$$\dot{\mathbf{v}} = \mathbf{g} + \frac{\mathbf{T}}{m}, \tag{17b}$$

$$\dot{m} = -\eta \left\| \mathbf{T} \right\|_2, \tag{17c}$$

where $\mathbf{r} = [x, z]^T$ represents the position vector of the vehicle, $\mathbf{v} = [v_x, v_z]^T$ is the velocity vector, $\mathbf{g} = [0, -g_0]^T$ is the vector of gravity acceleration ($g_0 = 3.711\ m/s^2$ is the gravity acceleration on Mars), $\mathbf{T} = [T_x, T_z]^T$ represents the engine thrust vector, m is the vehicle mass, and η is a positive constant associated with the effective exhaust velocity of the rockets. The magnitude of the thrust vector $\mathbf{T}$ is bounded by a lower bound and an upper bound. Thus

$$T_{min} \leq \|T(t)\|_2 \leq T_{max}, \ \forall\, t \in [t_0, t_f], \tag{18}$$

where t_0 and t_f represent the starting time and the final time of the powered descent phase. According to the Pontryagin minimum principle, the fuel-optimal thrust magnitude $\|\mathbf{T}\|$ is supposed to have a bang-bang profile [1], which means $\|\mathbf{T}\|$ should be at either the upper bound or the lower bound. Thus, the bounded constraint on the control magnitude in (18) can be replaced by

$$\|T(t)\|_2 \in \{T_{min}, T_{max}\}, \ \forall\, t \in [t_0, t_f]. \tag{19}$$

To prevent the vehicle from touching the ground before landing, the glide-slope constraint is considered, which can be written as

$$\mathbf{r} \in C := \left\{ \mathbf{r} \in \mathbb{R}^2 : \frac{\mathbf{r}\cdot\mathbf{e}}{\|\mathbf{r}\|\,\|\mathbf{e}\|} \geq \cos\theta \right\}, \tag{20}$$

where $\mathbf{e}$ is a unit vector in the direction of the z-axis and θ denotes the maximum of the glide-slope angle. Meanwhile, due to the limited mass of fuel, the vehicle mass is subject to the constraint

$$m(t_f) \geq m_{dry}, \tag{21}$$

where m_{dry} represents the structural mass of the vehicle. In the FOPDG problem, the initial states and terminal states are specified as

$$m(t_0) = m_0, \quad \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \tag{22a}$$

$$\mathbf{r}(t_f) = \mathbf{0}, \quad \mathbf{v}(t_f) = \mathbf{0}. \tag{22b}$$

Then, the FOPDG problem can be summarized as

$$\min_{\mathbf{T}, t_f} \quad -m(t_f), \tag{23a}$$

$$\text{subject to} \quad \dot{\mathbf{r}} = \mathbf{v}, \tag{23b}$$

$$\dot{\mathbf{v}} = \mathbf{g} + \frac{\mathbf{T}}{m}, \tag{23c}$$

$$\dot{m} = -\eta\,\|\mathbf{T}\|_2, \tag{23d}$$

$$\|\mathbf{T}(t)\|_2 \in \{T_{min}, T_{max}\}, \ \forall t \in [t_0, t_f], \tag{23e}$$

$$\mathbf{r} \in C := \left\{ \mathbf{r} \in \mathbb{R}^2 : \frac{\mathbf{r}\cdot\mathbf{e}}{\|\mathbf{r}\|\,\|\mathbf{e}\|} \geq \cos\theta \right\}, \tag{23f}$$

$$m(t_f) \geq m_{dry}, \tag{23g}$$

$$m(t_0) = m_0, \quad \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \tag{23h}$$

$$\mathbf{r}(t_f) = \mathbf{0}, \quad \mathbf{v}(t_f) = \mathbf{0}. \tag{23i}$$

By applying the discretization technique, the above FOPDG problem can be reformulated as a QCQP problem. To be specific, the continuous trajectory is discretized into H nodes represented by $[x_i,\ y_i],\ i = 1, \cdots, H$, at each discrete node. In the first stage, the bang-bang control constraints (23e) are relaxed as the bounded constraints according to (18), and the equality constraints on $\mathbf{r}_f$ and $\mathbf{v}_f$ which are not satisfied by the initial guess are also relaxed by introducing new slack variables ζ_r and ζ_v, and the weighting factor μ_r and μ_v. Then according to the Euler discretization rule, the OCP in (23) can be reformulated as a QCQP problem for the first stage, expressed as

$$\min_{[T_1,\ldots,T_N], t_f} \quad -m_N + \mu_r \cdot \zeta_r^2 + \mu_v \cdot \zeta_v^2 \tag{24a}$$

$$\text{subject to} \quad \frac{x_{i+1} - x_i}{\Delta t} = v_{x_i}, \ \frac{z_{i+1} - z_i}{\Delta t} = v_{z_i}, \tag{24b}$$

$$\frac{v_{i+1} - v_i}{\Delta t} = \frac{T_{x_i}}{m_i}, \ \frac{v_{i+1} - v_i}{\Delta t} = \frac{T_{z_i}}{m_i} - g_0, \tag{24c}$$

$$\frac{m_{i+1} - m_i}{\Delta t} = -\eta T_i, \ T_i^2 = T_{x_i}^2 + T_{x_i}^2, \tag{24d}$$

$$z_i^2 \geq x_i^2 \cdot \cot^2(\theta), \tag{24e}$$

$$m_N \geq m_{dry}, \tag{24f}$$

$$m_1 = m_0, \quad \mathbf{r}_1 = \mathbf{r}_0, \quad \mathbf{v}_1 = \mathbf{v}_0, \tag{24g}$$

$$\mathbf{r}_H - \zeta_r = \mathbf{0}, \quad \mathbf{v}_H - \zeta_v = \mathbf{0}, \tag{24h}$$

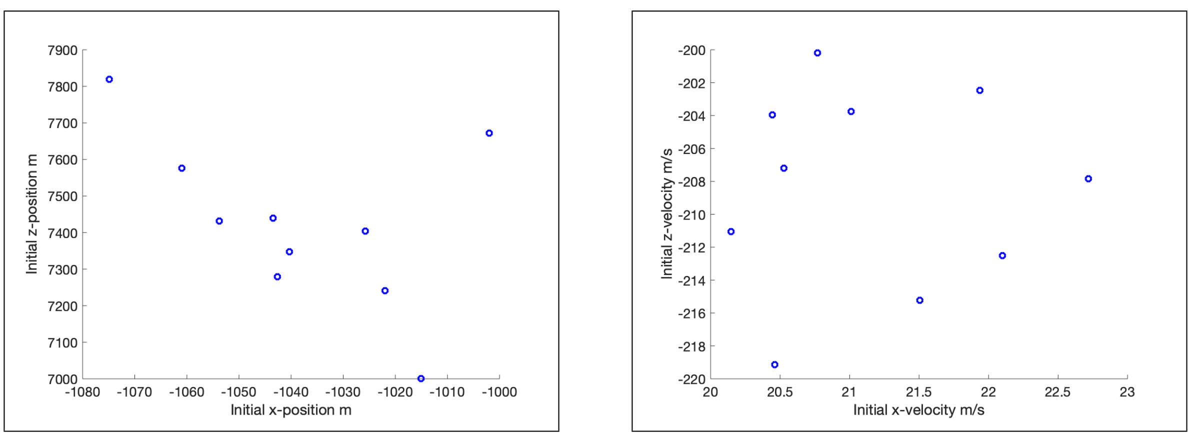
$$T_{min} \leq T_i \leq T_{max} \tag{24i}$$

where $i = 1, \cdots, H$, and Δt represents the uniform time step. In addition, in the second stage, the bang-bang control constraint (23e) on T_i can be expressed as a quadratic equality constraint, $(T_i - T_{min})(T_i - T_{max}) = 0$. Thus for the second stage, problem (23) can be reformulated as

$$\begin{aligned} \min_{[T_1,\dots,T_N],t_f} \quad & -m_N + \mu_r \cdot \zeta_r^2 + \mu_v \cdot \zeta_v^2 \\ \text{subject to} \quad & (24b), (24c), (24d), (24e), (24f), (24g), (24h), \\ & (T_i - T_{min})(T_i - T_{max}) = 0. \end{aligned} \tag{25}$$

Since the other constraints are expressed as linear or quadratic equalities/inequalities, the FOPDG is then reformulated as a QCQP problem. After that, the proposed multi-stage iterative algorithm is applied to present the simulation results. Here, to solve the SOCP problems in each iteration, a commercial solver Mosek [20] is used. In addition, the parameters in (24) and (25) are set as $g_0 = -3.7114\, m/s^2$, $m_0 = 51.1\ t$, $m_{dry} = 0.8m_0 = 40.88\ t$, $\eta = 4.53 \times 10^{-4}\ s/m$, $T_{max} = 640\, kN$, $T_{min} = 240\, kN$, $\theta = 86°$.

To verify the robustness and effectiveness of the proposed algorithm, 10 cases with random boundary conditions are solved via the multi-stage iterative algorithm and the NLP method, where the initial states and the fuel consumption of these cases are shown in Fig. 1 and Fig. 2. In Fig. 1, the ranges of the initial x-position, z-position, x-velocity and z-velocity are $[-1.1km, -1km]$, $[7km, 8km]$, $[20m/s, 23m/s]$, $[-220m/s, -200m/s]$, respectively. From Fig. 2, it can be found that all cases have a converged result using the multi-stage iterative algorithm and the NLP method. However, for each case, the fuel consumption amount from the solution of the multi-stage iterative algorithm is less than the corresponding one solved by the NLP method. In order to verify the accuracy of the proposed algorithm, the landing errors from both methods are compared and presented in Fig. 3a and Fig. 3b, where the green and red points denote the position/velocity errors from the proposed algorithm and the NLP method, respectively. Similarly, the terminal velocity errors from both methods are shown in Fig. 3b. It can be found that the proposed algorithm can achieve smaller position and velocity errors than those from the NLP method. In conclusion, compared with the NLP method, the proposed algorithm can lead to higher accuracy. To show more details of the multi-stage iterative algorithm and the NLP method, the complete solutions from both methods of one selected case are shown in Fig. 4 and 5, where the initial states of the powered descent phase are specified as $x(t_0) = -1025.8\ m$, $z(t_0) = 7403.9\, m$, $v_x(t_0) = 21.0\, m/s$, $v_z(t_0) = -203.8\, m/s$.



(a) Initial position for 10 cases **(b) Initial velocity for 10 cases**

Fig. 1 Initial position and velocity for 10 cases

As shown in Fig. 4a, for the multi-stage iterative algorithm, the terminal mass of the landing vehicle is 43.33 tons, which indicates that the fuel consumption is 8.08 tons, and it takes 22.0 second for the multi-stage iterative algorithm to converge. While for the NLP solver, the fuel consumption is 8.14 tons and the computational time for the NLP method is 2.2 seconds. In Fig. 4b, the thrust magnitude provided by the proposed algorithm and the NLP solver is presented, where the green, black, and blue curves represent the thrust magnitude, thrust components along the x-axis and the z-axis, respectively. From the comparative results of the demonstration case, it can be observed that the thrust magnitude obtained from the proposed multi-stage iterative algorithm is an exact bang-bang curve, while in the NLP result, there is a slope between the upper bound and the lower bound. Figure 5 demonstrates the history of the position

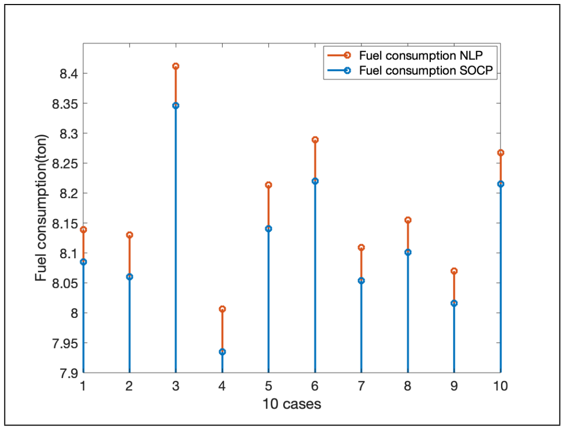
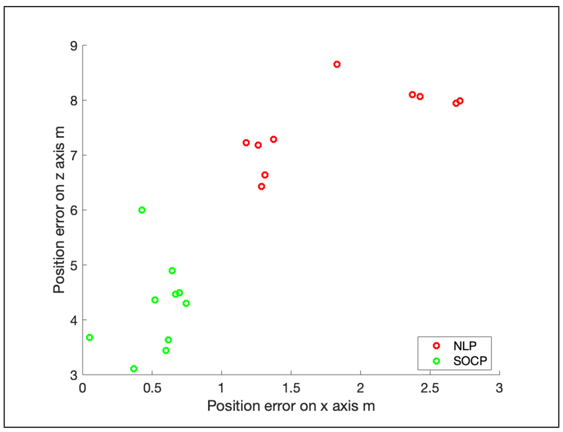
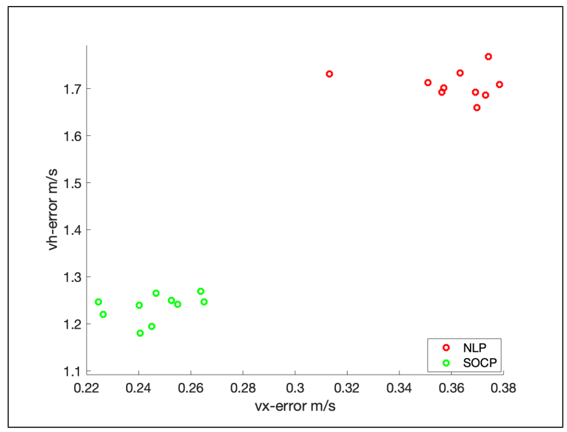


Fig. 2 Fuel consumption for 10 cases



(a) Position errors for 10 cases



(b) Velocity errors for 10 cases

Fig. 3 Position and velocity errors for 10 random cases

and the velocity in the powered descent phase, which indicates that the position and velocity from two methods share similar curves. Therefore, we can conclude that the multi-stage iterative algorithm can achieve more precise landing compared with the NLP method.

All in all, from the comparative simulation results, it can be concluded that the proposed multi-stage iterative algorithm has computational advantages in terms of accuracy, robustness, and cost value when solving Bang-bang OCPs.

V. Conclusion

In this paper, a multi-stage iterative algorithm is proposed to solve the bang-bang optimal control problem. Effectiveness and robustness are verified by applying the proposed algorithm to the fuel-optimal powered descent guidance problem. Simulation results show that for the bang-bang optimal control, the proposed method can find a local optimum with the controls being exact bang-bang curves. Moreover, by comparing with the optimized solutions obtained from the commercial nonlinear programming solver, it is verified that the proposed multi-stage iterative algorithm has computational advantages in terms of accuracy, robustness, and cost value when solving the bang-bang optimal control problem.

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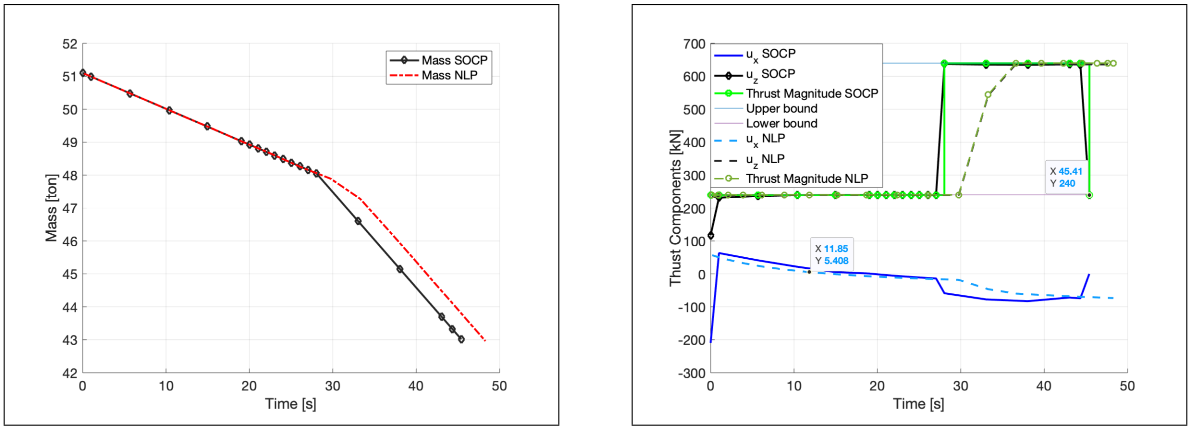
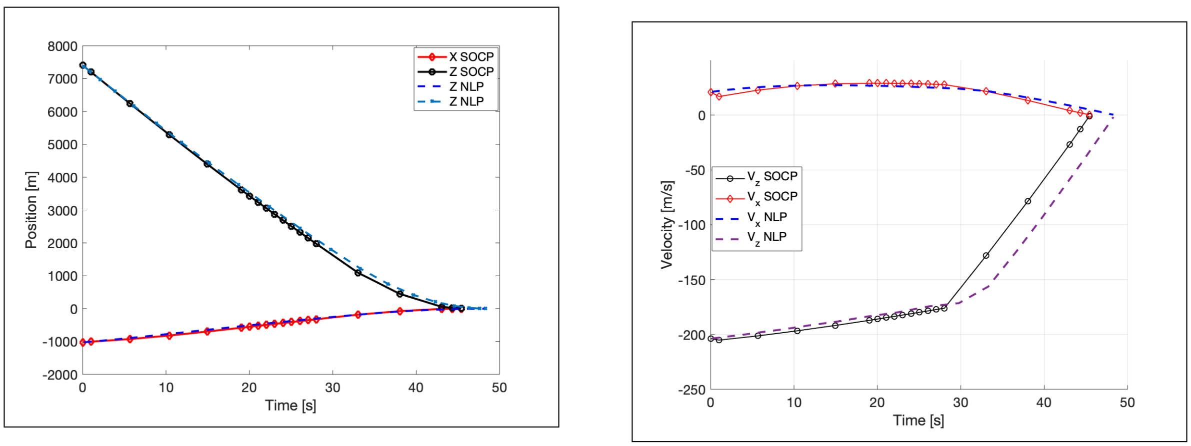


(a) Optimized total mass **(b) Optimized thrust magnitude**

Fig. 4 Optimized Thrust and mass from the multi-stage iterative algorithm



(a) Optimized trajectory from the multi-stage iterative algorithm **(b) Optimized velocities from the multi-stage iterative algorithm**

Fig. 5 Optimized trajectory and velocity

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