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Mixed-Input Learning for Multi-point Landing Guidance with Hazard Avoidance Part I: Offline Mission Planning based on Multi-Stage Optimization

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This paper proposes a multi-stage optimization framework based on iterative second-order cone programming (SOCP) to solve the three-dimensional (3D) multi-point landing guidance (MLG) problem with hazard avoidance. The approach is used to generate the offline optimal trajectories for database construction in Part II of this paper, it aims to select a safe landing point while finding an optimal path to the selected landing point with minimum fuel consumption. First, by introducing binary variables associated with quadratic constraints, the MLG problem with hazard avoidance is equivalently reformulated as a quadratically constrained quadratic programming (QCQP) problem. Next, to solve the reformulated QCQP problem, a multi-stage optimization framework, which is combined with the relaxation technique, is introduced. The proposed method includes two main stages. In the first stage, the reformulated problem is relaxed into a nonconvex QCQP problem via ignoring constraints related to the binary variables, which can be solved by the proposed iterative second-order cone programming (SOCP) with random initial guess. Via solving the relaxed QCQP problem with proposed iterative SOCP, the initial guess for the second phase is generated. In the second phase, with the generated initial guess in the first phase, the proposed iterative SOCP can find the local minimum for the equivalently reformulated QCQP problem. Finally, the effectiveness of the proposed method is verified via numerical simulations.

I. Introduction

With the development of terrestrial techniques for terrain relative navigation (TRN), the multi-point landing guidance (MLG) is getting increasing attention. Although, the TRN technology is still under development, extensive studies have been conducted on the map-based navigation to achieve precision landing with hazard avoidance [1]. However, the next-generation landing missions require an onboard evaluation of tens-to-hundreds of pre-specified landing sites, as well as the generation of an optimal trajectory at the same time, which makes MLG a challenging task [2].

In the traditional powered descent guidance problem, the target landing site is considered to be a fixed point, and the objective is to design a fuel-optimal trajectory that takes into account the vehicle's dynamics and mission constraints. Extensive approaches have been developed for the optimal powered descent guidance problem, which can be classified as direct and indirect methods. In the direct methods, the continuous states and controls are discretized, and then the continuous optimal control problems can be formulated as Nonlinear Programming (NLP) problems, which can be solved via different optimization algorithms. However, despite the fact that direct approaches have been used to solve a wide range of optimal control problems, it usually requires a good initial guess [3]. In addition, relaxation and convexification techniques have been adopted in the direct methods [4–6]. However, the traditional algorithms for solving NLP cannot handle binary variables.

On the other hand, indirect approaches convert an optimal control problem into a two-point boundary-value problem (TPBVP) based on Hamiltonian and first-order necessary conditions. To solve the resulted TPBVPs, shooting [7, 8] and

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approximation [9, 10] methods have been developed. Whereas in general, the resulting TPBVPs in an indirect method tend to be highly sensitive to the adjoint variables, which cannot guarantee convergence in general cases.

Instead of targeting a fixed landing point, the MLG problem requires to select a landing point from a set of feasible options within a specified landing region. Several approaches have been developed for landing point selection. For example, work in [2] selects the optimal landing site from hundreds of candidates by simply comparing the propellant mass for each target. A synthetic landing area assessment criterion is proposed in [11], and the best landing site is selected by evaluating terrain safety, fuel consumption, and touchdown performance during the descent phase. However, it is time consuming to evaluate all options when a great number of landing points are available. This paper proposes a multi-stage optimization framework to select a landing point while finding an optimal path leading to the selected landing point with minimum fuel consumption. In addition, from the existing analysis, the optimal control profile for the fuel-optimal powered descent guidance is proved to have a bang-bang curve [12]. Therefore, the optimal control profile is constrained to yield a bang-bang profile.

By expressing the binary constraints as quadratic equality constraints and employing the discretization techniques, the MLG problem can be formulated as a nonconvex quadratically constrained quadratic programming (QCQP) problem. Our previous work in [13] has developed a unified algorithm based on iterative second-order cone programming (SOCP) to solve bang-bang optimal control problems. In this paper, we extend the multi-stage optimization framework to the MLG problem. To obtain robust convergence of the iterative SOCP algorithm under random initial guess, the multi-stage optimization framework is combined with the relaxation technique to achieve robust convergece. To be specific, in the first stage, all the binary constraints are relaxed as upper and lower bounds, and the reformulated problem in the first stage is solved via the iterative SOCP algorithm. The solution from the first stage is used as an initial guess for the second stage, where the binary constraints of the original problem are reconsidered to obtain the final solution. Simulation results are provided to verify the effectiveness of the proposed method. The approach is used to generate the offline optimal trajectories for database construction in Part II of this paper [14].

This paper is organized as follows. Section II introduces the problem formulation. The proposed multi-stage optimization framework is detailed in Section III. After that, numerical simulations are provided and analyzed in Section IV. Conclusions are addressed in Section V.

II. Problem Formulation

For the 3D MLG problem with hazard avoidance, the landing vehicle is required to select a landing point from a target region containing hazards and then guide the vehicle to the selected landing point with high precision. Motion during the MLG is analyzed in a Cartesian coordinate system with the origin fixed to the surface of Mars. Without loss of generality, we assume the landing vehicle can be treated as a point mass and the origin of the Cartesian coordinate is located at the center of the target region. The 3D motion equations of the landing vehicle are expressed as

$$\dot{\mathbf{r}} = \mathbf{v}, \tag{1a}$$

$$\dot{\mathbf{v}} = \mathbf{g} + \frac{\mathbf{T}}{m}, \tag{1b}$$

$$\dot{m} = -\eta \left\|\mathbf{T}\right\|_2, \tag{1c}$$

where $\mathbf{r} = [x, y, z]^T$ represents the position vector, $\mathbf{v} = [v_x, v_y, v_z]^T$ is the velocity vector, $\mathbf{g} = [0, 0, -g_0]^T$ is the gravitational acceleration vector ($g_0 = 3.711\ m/s^2$ is the gravity acceleration on Mars), $\mathbf{T} = [T_x, T_y, T_z]^T$ denotes the vector of the engine thrust, m is the launch vehicle mass, and η is a positive constant associated with the fuel consumption rate of the rockets. T_{max} and T_{min} represents the lower bound and upper bound of the thrust magnitude $\mathbf{T}$. Thus,

$$T_{min} \le \|\mathbf{T}\|_2 \le T_{max},\ \forall t \in [t_0, t_f], \tag{2}$$

where t_0 and t_f are the starting and final time of the powered descent landing, respectively. According to the Pontryagin's maximum principle, the fuel-optimal thrust magnitude $\|\mathbf{T}\|$ is supposed to be at either the upper bound T_{max} or the lower bound T_{min}, which is a bang-bang profile [12]. Thus, the upper and lower bound constraints on the thrust magnitude in (2) can be written as

$$\|\mathbf{T}\|_2 \in \{T_{min}, T_{max}\},\ \forall t \in [t_0, t_f]. \tag{3}$$

To avoid the vehicle hitting the ground before landing, another state constraint, the glide slope constraint is considered, which can be written as

$$\mathbf{r} \in C := \left\{ \mathbf{r} \in \mathbb{R}^3 : \frac{\mathbf{r} \cdot \mathbf{e}}{\|\mathbf{r}\| \, \|\mathbf{e}\|} \geq \cos\theta \right\}, \tag{4}$$

where $\mathbf{e}$ is a unit vector pointing to the direction of the z-axis and θ is the maximum value of the glide slope angle. Meanwhile, due to the limitation on the fuel mass, the vehicle mass is constrained by

$$m(t_f) \geq m_{dry}, \tag{5}$$

where m_{dry} denotes the empty mass of the vehicle. In the MLG problem, the boundary constraints on the initial and terminal states are specified as

$$m(t_0) = m_0, \quad \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \quad \mathbf{v}(t_f) = 0 \tag{6}$$

Besides, to guarantee that the selected landing point locates in the target region, we assume that there are n potential landing points in the target region, and for the ith potential landing point, its position vector is expressed as $\mathbf{r}_{fi}$. Then, via introducing n binary variables c_i, the constraints on the terminal states can be expressed as

$$\sum_{i=1}^{n} c_i = 1, \tag{7a}$$

$$\sum_{i=1}^{n} c_i(\mathbf{r}_f - \mathbf{r}_{fi}) = 0, \tag{7b}$$

$$c_i(c_i - 1) = 0, i = 1, \ldots, n \tag{7c}$$

The objective of the MLG problem is to find the optimal landing point among a series of candidate points, and at the same time, design the fuel-optimal trajectory. Thus, the MLG problem can be summarized as

$$\min_{\mathbf{T}, t_f} \quad -m(t_f) + \sum_{i=1}^{n} c_i J(\mathbf{r}_{fi}), \tag{8a}$$

$$\text{subject to} \quad \dot{\mathbf{r}} = \mathbf{v}, \tag{8b}$$

$$\dot{\mathbf{v}} = \mathbf{g} + \frac{\mathbf{T}}{m}, \tag{8c}$$

$$\dot{m} = -\eta \, \|\mathbf{T}\|_2 \,, \tag{8d}$$

$$\|\mathbf{T}\|_2 \in \{T_{min}, T_{max}\}, \ \forall t \in [t_0, t_f], \tag{8e}$$

$$\mathbf{r} \in C := \left\{ \mathbf{r} \in \mathbb{R}^2 : \frac{\mathbf{r} \cdot \mathbf{e}}{\|\mathbf{r}\| \, \|\mathbf{e}\|} \geq \cos\theta \right\}, \tag{8f}$$

$$m(t_f) \geq m_{dry}, \tag{8g}$$

$$m(t_0) = m_0, \quad \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \quad \mathbf{v}(t_f) = 0, \tag{8h}$$

$$\sum_{i=1}^{n} c_i(\mathbf{r}_f - \mathbf{r}_{fi}) = 0, \sum_{i=1}^{n} c_i = 1, c_i(c_i - 1) = 0, i = 1, \ldots, n, \tag{8i}$$

where $J(\mathbf{r}_{fi})$ is the estimated extra cost for the ith pre-specified landing site. Specifically, significantly high extra costs are assigned for the potential landing sites in the hazard zones in order to guarantee a safe landing.

And then the above mentioned MLG problem (8) can be rewritten as a QCQP problem via applying the discretization technique. For the sake of clarity, the continuous trajectory is discretized into N nodes, represented by $[x_k, \ y_k, \ z_k], \ k =$

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$1, \cdots, N$ at each node. Then the MLG in (8) can be transformed into a QCQP problem, expressed as

$$\min_{[T_1,\ldots,T_N],t_f} \quad -m_N + \sum_{i=1}^{n} c_i J(\mathbf{r}_{fi}) \tag{9a}$$

$$\text{subject to} \quad \frac{x_{k+1}-x_k}{\Delta t} = v_{x_k},\ \frac{y_{k+1}-y_k}{\Delta t} = v_{y_k},\ \frac{z_{k+1}-z_k}{\Delta t} = v_{z_k}, \tag{9b}$$

$$\frac{v_{x,k+1}-v_{x,k}}{\Delta t} = \frac{T_{x_k}}{m_k},\ \frac{v_{y,k+1}-v_{y,k}}{\Delta t} = \frac{T_{y_k}}{m_k},\ \frac{v_{z,k+1}-v_{z,k}}{\Delta t} = \frac{T_{z_k}}{m_k} - g_0, \tag{9c}$$

$$\frac{m_{k+1}-m_k}{\Delta t} = -\eta T_k,\ T_k^2 = T_{x_k}^2 + T_{y_k}^2 + T_{z_k}^2, \tag{9d}$$

$$(z_k - z_f)^2 \geq (x_k - x_f)^2 \cdot \cot^2(\theta) + (y_k - y_f)^2 \cdot \cot^2(\theta), \tag{9e}$$

$$m_N \geq m_{dry}, \tag{9f}$$

$$m_1 = m_0,\quad \mathbf{r}_1 = \mathbf{r}_0,\quad \mathbf{v}_1 = \mathbf{v}_0,\quad \mathbf{v}_N = 0, \tag{9g}$$

$$\sum_{i=1}^{n} c_i(\mathbf{r}_N - \mathbf{r}_{fi}) = 0, \sum_{i=1}^{n} c_i = 1, \tag{9h}$$

$$c_i(c_i - 1) = 0, \tag{9i}$$

$$(T_k - T_{min})(T_k - T_{max}) = 0. \tag{9j}$$

where $i = 1, \cdots, n$, $k = 1, \cdots, N$, and Δt represents the uniform time step. Next, the multi-stage optimization framework is described, which is applied to solve the formulated QCQP problem.

III. Multi-Stage Optimization Framework with Iterative SOCP

A. Multi-stage Iterative Algorithm

For the MLG problem, to obtain robust convergence of the proposed algorithm under random initial guess, a multi-stage optimization framework is proposed, which includes two stages. In the first stage, only the fuel consumption is considered in the objective function. In addition, the quadratic equality constraints on the bang-bang control profile and the multiple landing points are relaxed as upper and lower bound inequality constraints in the first stage. Accordingly, problem in (9) is cast as

$$\min_{[T_1,\ldots,T_N],t_f} \quad -m_N \tag{10a}$$

$$\text{subject to} \quad \frac{x_{k+1}-x_k}{\Delta t} = v_{x_k},\ \frac{y_{k+1}-y_k}{\Delta t} = v_{y_k},\ \frac{z_{k+1}-z_k}{\Delta t} = v_{z_k}, \tag{10b}$$

$$\frac{v_{x,k+1}-v_{x,k}}{\Delta t} = \frac{T_{x_k}}{m_k},\ \frac{v_{y,k+1}-v_{y,k}}{\Delta t} = \frac{T_{y_k}}{m_k},\ \frac{v_{z,k+1}-v_{z,k}}{\Delta t} = \frac{T_{z_k}}{m_k} - g_0, \tag{10c}$$

$$\frac{m_{k+1}-m_k}{\Delta t} = -\eta T_k,\ T_k^2 = T_{x_k}^2 + T_{y_k}^2 + T_{z_k}^2, \tag{10d}$$

$$(z_k - z_f)^2 \geq (x_k - x_f)^2 \cdot \cot^2(\theta) + (y_k - y_f)^2 \cdot \cot^2(\theta), \tag{10e}$$

$$m_N \geq m_{dry}, \tag{10f}$$

$$m_1 = m_0,\quad \mathbf{r}_1 = \mathbf{r}_0,\quad \mathbf{v}_1 = \mathbf{v}_0,\quad \mathbf{v}_N = 0, \tag{10g}$$

$$\sum_{i=1}^{n} c_i(\mathbf{r}_N - \mathbf{r}_{fi}) = 0, \sum_{i=1}^{n} c_i = 1, \tag{10h}$$

$$0 \leq c_i \leq 1, i = 1, \ldots, n, \tag{10i}$$

$$T_{min} \leq T_i \leq T_{max} \tag{10j}$$

The relaxed problem in the first stage is then solved by the iterative SOCP algorithm [13], and more details will be introduced in the next subsection.

In the first stage, due to the relaxed binary constraints, the terminal point is not necessarily to be exact one of the candidate landing sites. Based on the solution of the first stage, to generate a more reasonable initial guess for the

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second stage, the landing point that is closest to the terminal point in the first stage that is also outside of the avoidance zones, is selected, and the values of $c_i, i = 1, \cdots, n$ are assigned accordingly. The solution from the first stage, as well as the allocated $c_i, i = 1, \cdots, n$, are combined as the initial guess of the second stage.

For the second stage, both the fuel consumption and the extra costs for the potential landing sites are considered in the objective function. Moreover, the binary constraints on the bang-bang profile and the multiple landing points are reconsidered. To improve the precision of the discretized trajectory, the time intervals are considered as additional variables. And then the problem can be formulated as

$$\min_{[T_1,\ldots,T_N],[\Delta t_1,\ldots,\Delta t_N]} \quad -m_N + \sum_{i=1}^{n} c_i J(\mathbf{r}_{fi}) \tag{11a}$$

$$\text{subject to} \quad \frac{x_{k+1}-x_k}{\Delta t_k} = v_{x_k}, \ \frac{y_{k+1}-y_k}{\Delta t_k} = v_{y_k}, \ \frac{z_{k+1}-z_k}{\Delta t_k} = v_{z_k}, \tag{11b}$$

$$\frac{v_{x,k+1}-v_{x,k}}{\Delta t_k} = \frac{T_{x_k}}{m_k}, \ \frac{v_{y,k+1}-v_{y,k}}{\Delta t_k} = \frac{T_{y_k}}{m_k}, \ \frac{v_{z,k+1}-v_{z,k}}{\Delta t_k} = \frac{T_{z_k}}{m_k} - g_0, \tag{11c}$$

$$\frac{m_{k+1}-m_k}{\Delta t_k} = -\eta T_k, \ T_k^2 = T_{x_k}^2 + T_{y_k}^2 + T_{z_k}^2, \tag{11d}$$

$$(z_k - z_f)^2 \geq (x_k - x_f)^2 \cdot \cot^2(\theta) + (y_k - y_f)^2 \cdot \cot^2(\theta), \tag{11e}$$

$$m_N \geq m_{dry}, \tag{11f}$$

$$m_1 = m_0, \quad \mathbf{r}_1 = \mathbf{r}_0, \quad \mathbf{v}_1 = \mathbf{v}_0, \quad \mathbf{v}_N = 0, \tag{11g}$$

$$\sum_{i=1}^{n} c_i(\mathbf{r}_N - \mathbf{r}_{fi}) = 0, \sum_{i=1}^{n} c_i = 1, \tag{11h}$$

$$c_i(c_i - 1) = 0, \tag{11i}$$

$$(T_k - T_{min})(T_k - T_{max}) = 0. \tag{11j}$$

Using the iterative SOCP algorithm again, the formulated problem in the second stage, which includes all constraints of the original problem, is resolved. The framework of the proposed multi-stage optimization framework is presented in Table 1.

Table 1 Flowchart of the Multi-stage Optimization Framework

STAGE 1
Input: Random initial guess $\bar{\mathbf{x}}^0$
Output: Thrusts $[\mathbf{T}_1, \ldots, \mathbf{T}_N]$, final time t_f
Begin: Reformulate the original QCQP problem (9) into relaxed problem (10)
and apply the iterative SOCP algorithm to solve it, until the converged solution is obtained.
STAGE 2
Input: The formulated problem (11)
solution from STAGE 1 and c_i allocation according to the closest safe landing site as the initial guess,
Output: Thrusts $[\mathbf{T}_1, \ldots, \mathbf{T}_N]$, time intervals $[\Delta t_1, \ldots, \Delta t_N]$
Begin: Apply the iterative SOCP algorithm to solve (9) with binary constraints,
until the converged solution is obtained.

B. The Iterative SOCP Algorithm

In this subsection, the iterative SOCP algorithm proposed in our previous work [13], is introduced and applied to the multi-point FOPDG problem.

In the above problem (10) and (11), by introducing extra variables and quadratic equality constraints, the objective function and constraints can be converted into a quadratic form[15]. Furthermore, when an equality constraint is taken into consideration, it can be equivalently replaced by two inequality constraints. Then the discretized problem in (9) and

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(10) can be formulated as an inhomogeneous QCQP problem,

$$
\begin{aligned}
\min_{\bar{\mathbf{y}}} \quad & \frac{1}{2}\mathbf{y}^T\mathbf{P}_0\mathbf{y} + \mathbf{q}_0^T\mathbf{y}, \\
\text{s.t.} \quad & \frac{1}{2}\mathbf{y}^T\mathbf{P}_j\mathbf{y} + \mathbf{q}_j^T\mathbf{y} + s_j \leq 0, \ \ j = 1, ..., H,
\end{aligned} \tag{12}
$$

where $\mathbf{y} \in \mathbb{R}^m$ denotes the unknown vector to be optimized, $P_j \in \mathbb{R}^{m\times m}$ are symmetric matrices, $\mathbf{q}_j \in \mathbb{R}^m$ are known vectors and $s_j \in \mathbb{R}$ represent the given constants, H is the number of the inequality constraints.

The inhomogeneous QCQP problem presented in (10) and (11) are then recast as a homogeneous rank-one constrained problem. Following that, By introducing a new variable $\mathbf{x} = [\mathbf{y}^T, 1]^T$, the inhomogeneous QCQP problem (12) can be equivalently transformed as a homogeneous QCQP

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^{m+1}} \quad & \mathbf{x}^T\mathbf{A}_0\mathbf{x}, \\
\text{s.t.} \quad & \mathbf{x}^{\mathrm{T}}\mathbf{A}_j\mathbf{x} + s_j \leq 0, \ \ j = 1, ..., H,
\end{aligned} \tag{13}
$$

where $\mathbf{A}_j = \begin{bmatrix} \mathbf{P}_j & \mathbf{q}_j/2 \\ \mathbf{q}_j^T/2 & 0 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}$. According to the equation $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{Tr}(\mathbf{A}\mathbf{x}\mathbf{x}^T)$, where $\mathbf{Tr}(\cdot)$ represents the trace of a matrix, by denoting $\mathbf{X} = \mathbf{x}\mathbf{x}^T \in \mathbb{R}^{(m+1)\times(m+1)}$, problem (13) can be expressed as

$$
\begin{aligned}
\min_{\mathbf{X},\mathbf{x}} \quad & \mathbf{Tr}(\mathbf{A}_0\mathbf{X}), \\
\text{s.t.} \quad & \mathbf{Tr}(\mathbf{A}_j\mathbf{X}) + s_j \leq 0, \ \ j = 1, ..., H, \\
& \mathbf{X} = \mathbf{x}\mathbf{x}^T,
\end{aligned} \tag{14}
$$

where the enty $X(i, j) = x(i)x(j)$ in matrix $\mathbf{X}$. Subsequently, the constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ can be equivalently converted to a rank-one constraint, $\text{rank}(\mathbf{X}) \leq 1$, together with a semidefinite constraint $\mathbf{X} \succeq \mathbf{0}$. Therefore, problem (14) can be further transformed into a rank-one constrained SDP problem as

$$
\begin{aligned}
\min_{\mathbf{X}} \quad & \mathbf{Tr}(\mathbf{A}_0\mathbf{X}), \\
\text{s.t.} \quad & \mathbf{Tr}(\mathbf{A}_j\mathbf{X}) + s_j \leq 0, \ \ j = 1, ..., H, \\
& \text{rank}(\mathbf{X}) \leq 1, \\
& \mathbf{X} \succeq \mathbf{0}.
\end{aligned} \tag{15}
$$

Hence, without loss of generality, a general QCQP problem is equivalently reformulated as a rank-one constrained SDP problem. And yet, the rank-one constraint in the problem (15) is still nonconvex. Therefore, solving large-scale rank-constrained SDP problems is always time-consuming.

For the purpose of enhancing the computational efficiency, in the second part, the semidefinite constraint is decomposed into a collection of second-order cone constraints. Here, a 2×2 sub-matrix of $\mathbf{X}$ is represented as

$$
\mathbf{X}_\beta = \begin{bmatrix} X(p,p) & X(p,q) \\ X(q,p) & X(q,q) \end{bmatrix},
$$

where β denotes an integer pair (p, q). Moreover, two sets of integer pairs are introduced, represented as $\mathcal{F}$ and $\mathcal{G}$, respectively,

$$
\mathcal{F} := \{(p,q) | q \neq m+1\},
$$

$$
\mathcal{G} := \{(p,q) | q = m+1\}.
$$

In problem (15), equations $\mathbf{x} = [\mathbf{y}^T, 1]^T$ and $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ imply that $X(m+1, m+1) = 1$. As a result, for a 2×2 sub-matrix $\mathbf{X}_\gamma$, where $\gamma \in \mathcal{G}$, we have

$$
\mathbf{X}_\gamma = \begin{bmatrix} X(p,p) & X(p,m+1) \\ X(m+1,p) & 1 \end{bmatrix}.
$$

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Theorem III.1 *[16] All of the* 2×2 *primal minors of a rank-one symmetric positive semidefinite matrix are equal to zero, indicating that all of its* 2×2 *sub-matrices are rank-one positive semidefinite matrices.*

According to Theorem III.1, for problem (15), the semidefinite constraint $\mathbf{X}_\gamma \geq \mathbf{0}$ can be recast as

$$X(p,p) - X(p,m+1)^2 \geq 0,$$
$$X(p,p) \geq 0,$$

which are second-order-cone and linear constraints. If non-zero elements only exist at the last row, last column and principal diagonal of the coefficient matrices, it is evident that only the semidefinite constraints on the sub-matrix $\mathbf{X}_\gamma$, where $\gamma \in \mathcal{G}$ need to be considered. And then $X(p,q)$ can be represented with a quadratic function as

$$X(p,q) = x(p)x(q) = \frac{1}{2}[(x(p)+x(q))^2 - x(p)^2 - x(q)^2]. \tag{16}$$

By introducing a new vector $\mathbf{z} = [z_1, \cdots, z_h]^T$, where $z_{i_h} = x(p) + x(q)$, $i_h = 1, \ldots, h$, the pair $(p,q) \in \mathcal{F}$, and h is the number of cross terms involved in the set $\mathcal{F}$, then (16) can be written as

$$2X(p,q) = (z_{i_h})^2 - X(p,p) - X(q,q). \tag{17}$$

Denote $\hat{\mathbf{x}} = [\mathbf{x}^T, \mathbf{z}^T, 1]^T \in \mathbb{R}^{(m+h+2)}$ and $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^T$, then $2\hat{X}(p,q) = \hat{X}(n+i_h, n+i_h) - \hat{X}(p,p) - \hat{X}(q,q)$, $i_h = 1, \ldots, h$. As the original vector $\mathbf{x}$ has now been extended to $\hat{\mathbf{x}}$, the coefficient matrices $\mathbf{A}_j$ can be reformulated as $\hat{\mathbf{A}}_j \in \mathbb{S}^{(m+h+2)\times(m+h+2)}$ such that $\mathbf{Tr}(\mathbf{A}_j\mathbf{X}) = \mathbf{Tr}(\hat{\mathbf{A}}_j\hat{\mathbf{X}})$, $j = 1, \ldots, H$. Besides, given that only non-zero elements in $\hat{\mathbf{A}}_j$, $j = 0, \ldots, H$, will be involved in the second-order cone constraints, we introduce the following definition.

Definition III.2 *Let* $\hat{\mathbf{A}}_t = \sum_{j=0}^{m} abs(\hat{\mathbf{A}}_j)$*, where* $abs(\hat{\mathbf{A}}_j)$ *denotes the element-wise absolute value of the matrix* $\hat{\mathbf{A}}_j$*, then we can define a set* $\hat{\mathcal{H}}$ *as*

$$\hat{\mathcal{H}} := \{(p,q) | p,q \in \{1, ..., m+h+2\} \;\&\; 1 \leq p < q \leq m+h+2\}.$$

Denote the kth entry in $\hat{H}$ *as* β_k*, where* $1 \leq k \leq K$, $K = \frac{(m+h+1)(m+h+2)}{2}$*. Then set* $\hat{\mathcal{K}}$ *is defined as*

$$\hat{\mathcal{K}} := \{\hat{\beta}_k | \hat{A}_t(p,q) \neq 0 \;\&\; \hat{\beta}_k \in \hat{\mathcal{H}}\},$$

where $\hat{Q}_t(p,q)$ *is the entry in pth row and qth column of matrix* $\hat{\mathbf{A}}_t$*. Similarly, we have*

$$\hat{\mathcal{F}} := \{\hat{\beta}_f | q \neq m+h+2 \;\&\; \hat{\beta}_f \in \hat{\mathcal{K}}\} = \emptyset,$$

$$\hat{\mathcal{G}} := \{\hat{\beta}_g | q = m+h+2 \;\&\; \hat{\beta}_g \in \hat{\mathcal{K}}\}.$$

And then, the rank-constrained SDP problem in (15) can be reformulated as a a rank-one constrained SOCP problem using the aforementioned definition.

$$\begin{aligned}
\min_{\hat{\mathbf{X}}} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_0\hat{\mathbf{X}}), \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_j\hat{\mathbf{X}}) + s_j \leq 0, \; j = 1, ..., H, \\
& \hat{\mathbf{X}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{X}(p,p) & \hat{X}(p,q) \\ \hat{X}(q,p) & \hat{X}(q,q) \end{bmatrix}, \; \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(m+i_h, 1) = \hat{X}(p,1) + \hat{X}(q,1), \; \forall \hat{\beta}_f \in \hat{\mathcal{F}}, i_h = 1, \ldots, h, \\
& \hat{X}(p,p) \geq 0, \; \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(p,p) - \hat{X}(p, m+n+2)^2 \geq 0, \; \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \text{rank}(\hat{\mathbf{X}}) \leq 1.
\end{aligned} \tag{18}$$

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Then, the rank contraint can be transformed into multiple rank-one constraints on the 2×2 principle submatrices of $\hat{\mathbf{X}}$ by applying decomposition techniques based on Theorem III.1. Therefore, problem (18) can be equivalently transformed into a new rank-one constrained SDP problem, represented by

$$
\begin{aligned}
\min_{\hat{\mathbf{X}}} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_0\hat{\mathbf{X}}), \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_j\hat{\mathbf{X}}) + c_j \leq 0,\ j = 1, ..., H, \\
& \hat{\mathbf{X}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{X}(p,p) & \hat{X}(p,q) \\ \hat{X}(q,p) & \hat{X}(q,q) \end{bmatrix},\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(m+i_h, 1) = \hat{X}(p,1) + \hat{X}(q,1),\ \forall \hat{\beta}_f \in \mathcal{F}, i_h = 1, \ldots, h, \\
& \hat{X}(p,p) \geq 0,\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(p,p) - \hat{X}(p, m+n+2)^2 \geq 0,\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \text{rank}(\hat{\mathbf{X}}_{\hat{\beta}_\mathbf{g}}) \leq 1.
\end{aligned} \tag{19}
$$

For each 2×2 submatrix $\hat{\mathbf{X}}_{\hat{\beta}_g}$, we define λ_1 and λ_2 as its two eigenvalues, and assume that $\lambda_1 \leq \lambda_2$. Correspondingly, there are two eigenvectors $\mathbf{v}^1_{\hat{\beta}_g}$ and $\mathbf{v}^2_{\hat{\beta}_g}$ which have

$$
\lambda_1 \mathbf{v}^1_{\hat{\beta}_g} = \hat{\mathbf{X}}_{\hat{\beta}_g} \mathbf{v}^1_{\hat{\beta}_g},\ \lambda_2 \mathbf{v}^2_{\hat{\beta}_g} = \hat{\mathbf{X}}_{\hat{\beta}_g} \mathbf{v}^2_{\hat{\beta}_g}. \tag{20}
$$

Due to the fact that $\hat{\mathbf{X}}_{\hat{\beta}_g}$ is a rank-one positive semidefinite matrix, indicating that $\lambda_2 \geq \lambda_1$ and $\lambda_1 = 0$, we have

$$
(\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{X}}_{\hat{\beta}_g} \mathbf{v}^1_{\hat{\beta}_g} = (\mathbf{v}^1_{\hat{\beta}_g})^T (\lambda_1 \mathbf{v}^1_{\hat{\beta}_g}) = 0. \tag{21}
$$

By introducing another new variable $r_{\hat{\beta}_g} \in \mathbb{R}$, the rank-one constraint $\text{rank}(\hat{\mathbf{X}}_{\hat{\beta}_g}) = 1$ can be reformulated as

$$
r_{\hat{\beta}_g} - (\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{X}}_{\hat{\beta}_g} \mathbf{v}^1_{\hat{\beta}_g} \geq 0, \tag{22}
$$

where $r_{\hat{\beta}_g} = 0$. However, the eigenvectors $\mathbf{v}^1_{\hat{\beta}_g}$ and $\mathbf{v}^2_{\hat{\beta}_g}$ can not be determined before obtaining the exact solution of $\hat{\mathbf{X}}_{\hat{\beta}_g}$.

For this reason, in the last step, the optimal eigenvectors are approached by gradually minimizing the independent variable $r_{\hat{\beta}_g}$. Thus, in order to accomplish this, in the new problem, the penalty term related to the rank-one constraint is minimized together with the original objective function. And the problem (19) can be reformulated as

$$
\begin{aligned}
\min_{\hat{\mathbf{X}}} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_0\hat{\mathbf{X}}) + \omega_l \sum_{\hat{\beta}_g \in \hat{\mathcal{G}}} r_{\hat{\beta}_g} \\
\text{s.t.} \quad & \mathbf{Tr}(\hat{\mathbf{A}}_j\hat{\mathbf{X}}) + c_j \leq 0,\ \ j = 1, ..., H, \\
& \hat{\mathbf{X}}_{\hat{\beta}_g} = \begin{bmatrix} \hat{X}(p,p) & \hat{X}(p,q) \\ \hat{X}(q,p) & \hat{X}(q,q) \end{bmatrix},\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(m+i_h, 1) = \hat{X}(p,1) + \hat{X}(q,1),\ \forall \hat{\beta}_f \in \mathcal{F},\ i_h = 1, \ldots, h, \\
& \hat{X}(p,p) \geq 0,\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& \hat{X}(p,p) - \hat{X}(p, n+1)^2 \geq 0,\ \forall \hat{\beta}_g \in \hat{\mathcal{G}}, \\
& r_{\hat{\beta}_g} - (\mathbf{v}^1_{\hat{\beta}_g})^T \hat{\mathbf{X}}_{\hat{\beta}_g} \mathbf{v}^1_{\hat{\beta}_g} \geq 0,\ \forall\, \hat{\beta}_g \in \hat{\mathcal{G}},
\end{aligned} \tag{23}
$$

where $\omega_l > 0$ denotes the weighting factor for $\hat{\beta}_g$ at the lth iteration. However, before finding $\hat{\mathbf{X}}_{\hat{\beta}_g}$ for all $\hat{\beta}_g \in \hat{\mathcal{G}}$, its corresponding eigenvectors $\mathbf{v}^1_{\hat{\beta}_g}$ can not be obtained. Therefore, an iterative framework based on SOCP is presented to solve the QCQP problem. The steps of the iterative SOCP are listed in Table 2.

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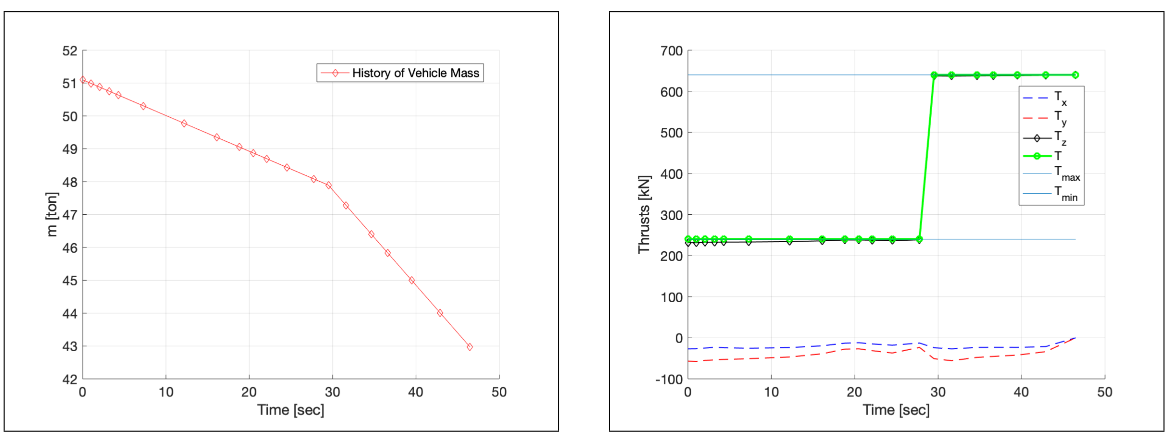
Table 2 Flowchart of the Iterative SOCP Algorithm

Input: $\mathbf{P}_j$, $\mathbf{q}_j$, s_j, $j = 0, 1, ..., H$, ω_l, $\hat{l}_{\max}$, and ϵ
Output: Unknown vector $\mathbf{y}$
begin:
1) Compute $\hat{\mathbf{A}}_j$, $j = 0, \ldots H$, according to the input
2) Calculate $\hat{\mathbf{X}}$ and $\mathbf{v}^1_{\hat{\beta}_g}$ with a random initial guess
3) **for** $l = 1, 2, ..., \hat{l}_{\max}$
4) Solve (23) to obtain solution $\hat{\mathbf{X}}$ and $r_{\hat{\beta}_g}$,
5) **If** $\sum_{\hat{\beta}_g \in \hat{\mathcal{G}}} r_{\hat{\beta}_g} \leq \epsilon$,
6) break;
7) **else**
8) Update $\mathbf{v}^1_{\hat{\beta}_g}$ from eigenvectors of $\hat{\mathbf{X}}_{\beta_g}$
9) **end if**
10) $l = l + 1$
11) **end for**

IV. Simulation Results

To verify the performance of the proposed multi-stage optimization framework, the numerical simulation results of the MLG problem are provided. Here, to solve the SOCP problems in each iteration, a commercial solver Mosek [17] is used. In the simulation case, 100 pre-specified landing sites are distributed evenly in the range of $X \sim [-2km, 2km]$ and $Y \sim [-2km, 2km]$. In addition, the parameters in (9) and (10) are set as $g_0 = -3.7114\, m/s^2$, $m_0 = 51.1\ t$, $m_{dry} = 0.8m_0 = 40.88\ t$, $\eta = 4.53 \times 10^{-4}\ s/m$, $T_{max} = 640\, kN$, $T_{min} = 240\, kN$, $\theta = 86°$ and the initial states of the powered descent phase are specified as $x(t_0) = -1025.75\ m$, $y(t_0) = -512.88\ m$, $z(t_0) = 7403.86\, m$, $v_x(t_0) = 21.01\, m/s$, $v_x(t_0) = 42.02\, m/s$, $v_z(t_0) = -203.75\, m/s$. In the simulation cases, we use the flatness of the terrain as the criteria for assigning the extra costs for all the potential landing points.



(a) Optimized total mass **(b) Optimized thrust magnitude**

Fig. 1 Optimized mass and thrust components

As shown in Fig. 1a, the terminal mass of the landing vehicle is 42.97 tons, which indicates that the fuel consumption is 8.13 tons, and it takes 71.0 second for the proposed method to converge. The topographic map and the optimal trajectories from stage 1 and stage 2 are shown in the Fig. 3a and Fig. 3b, respectively. From the optimized trajectories of the simulation case, it can be observed that in the first stage, the terminal point of the trajectory does not overlap with any potential landing points. While in the second stage, due to the binary constraints on the terminal states, the optimal trajectory ends at exactly one of the candidate landing sites. In Fig. 1b, the thrust magnitude provided by the

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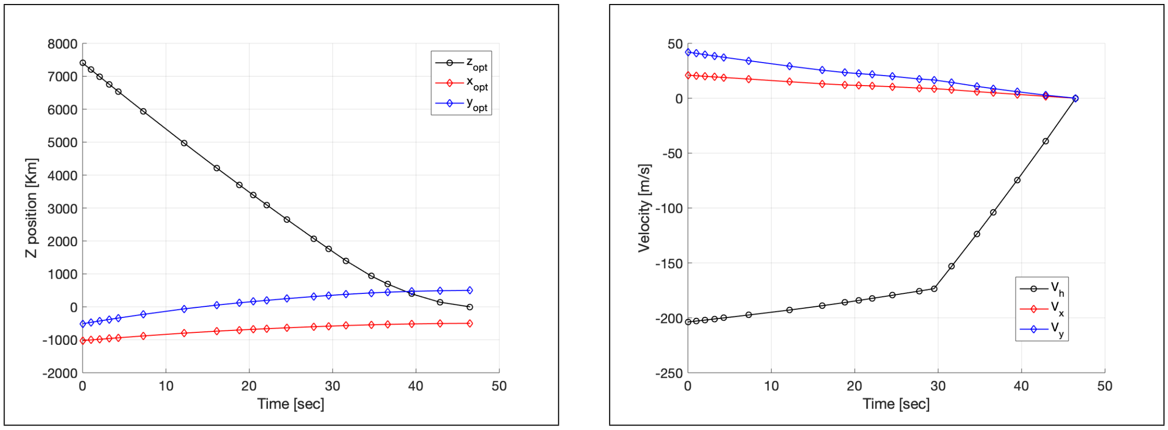
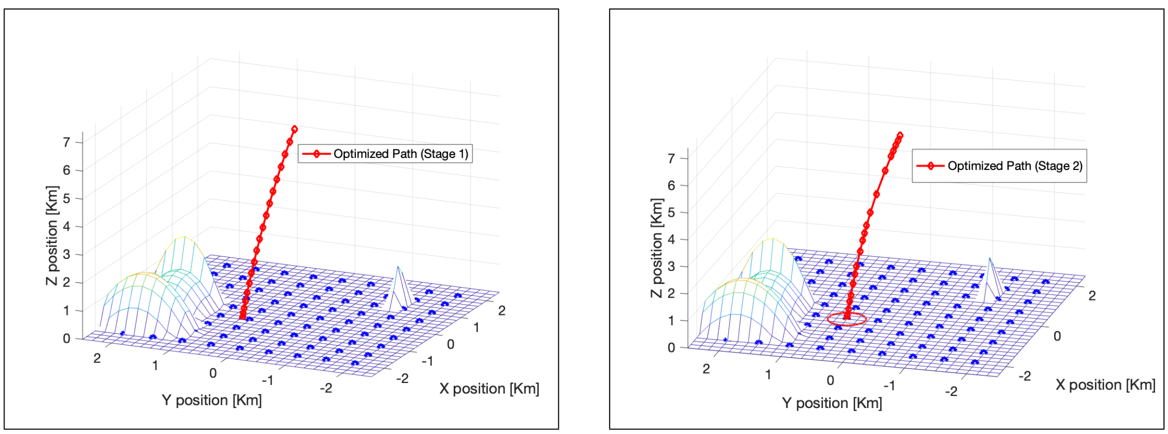


(a) Optimized trajectory from the multi-stage iterative algorithm

(b) Optimized velocities from the multi-stage iterative algorithm

Fig. 2 Optimized trajectory and velocity



(a) Optimized trajectory from stage 1

(b) Optimized trajectory from stage 2

Fig. 3 Optimized trajectory from the multi-stage iterative algorithm

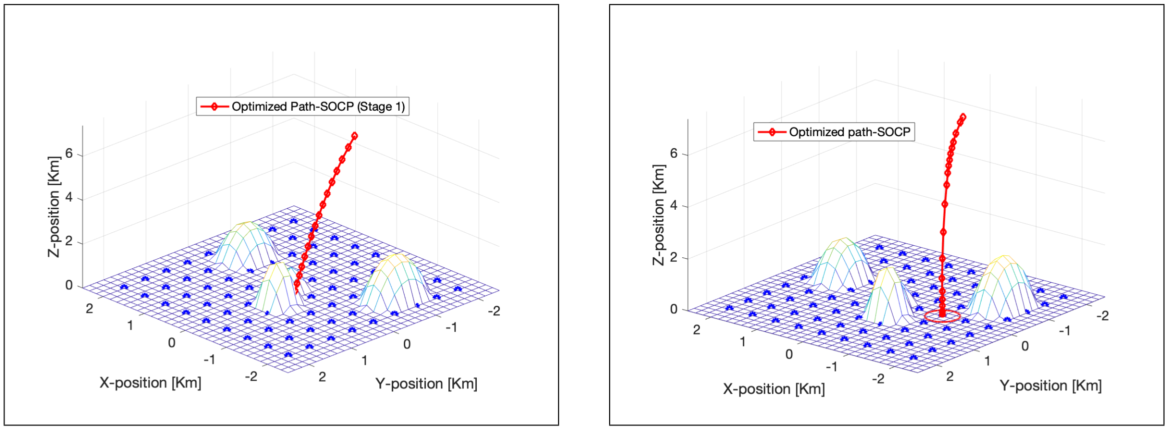
proposed algorithm is presented, where the green, black, blue and red curves represent the thrust magnitude, thrust components along the x-axis, the y-axis and the z-axis, respectively. It shows that the thrust magnitude obtained is an exact bang-bang curve. More details of the position history and the velocity history in the powered descent phase are demonstrated in Fig. 2.

Besides, the optimized trajectories of another case are provided in Figs. 4a and 4b. From this case, it can be observed that in the first stage, since only the fuel consumption is considered in the objective function, the landing vehicle finally landed in an area with uneven terrain. Whereas, in the second stage, due to the consideration of the extra costs from the selected landing points in the first phase, the trajectory switched to a flat landing site.

In summary, from the simulation results, it can be concluded that the proposed multi-stage optimization framework can effectively solve the MLG problem with binary decision variables. Extensive simulation cases have been generated for constructing a database for Part II of this topic [14].

V. Conclusion

In this paper, a multi-stage optimization framework is developed to solve the three dimensional multi-point landing guidance (MLG) problem. Simulation results show that for the MLG problem, the proposed method can find a bang-bang optimal control solution while avoiding hazard zones. The proposed methods have been used for offline database generation in Part II of this topic.



(a) Optimized trajectory from stage 1

(b) Optimized trajectory from stage 2

Fig. 4 Optimized trajectory from the multi-stage iterative algorithm

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